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Multimatrix models induced by group extensions

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Abstract. Multimatrix models for which the index set has a group structure and the interaction obeys a 'zero curvature' condition can be deformed related to central extensions of this group. The deformed multimatrix models lead to statistical systems defined on random graphs with a topological action. It is shown, how these topological theories on graphs can be used to weight graphs according to topological conditions.

1. Introduction

Hermitian multimatrix models generate summations over graphs with a weight which can be interpreted as the partition function of a statistical model on this graph (see e.g. [1–7]). Taking a special limit of the coupling constants towards critical points, and for the size N of the matrices to infinity—the so-called 'double scaling limit' [8]—leads to theories which describe matter coupled to two-dimensional quantum gravity. The type of matter (its central charge as well as the operator content of the fields) in general depends on the multimatrix model.

The special relation between Hermitian matrix integration and two-dimensional quantum gravity stems from the fact that the perturbation expansion of matrix models leads to the summation over Feynman graphs which have a natural interpretation as triangulations of two-dimensional surfaces, at least if restricted to Feynman graphs without self-loops or multilines (several lines connecting the same pair of vertices), which can usually be realized by a renormalization of the coupling constants. There have been attempts to use a similar formalism to generate triangulations of higher dimensional manifolds leading to regularized versions of higher dimensional quantum gravity. This can be achieved by using integrals over multi-indexed (tensor) objects [9–11]. One problem with this approach is that one has to fix the topology of the triangulations as otherwise the number of triangulations grows faster than exponential with the volume. The size of the multi-indexed quantities generalizing Hermitian matrices can be used to weight graphs according to the Euler number. This might not be enough, however, to guarantee the exponential growth. For odd dimensional manifolds without boundary the situation is even worse as the Euler number is always zero, a condition which has to be fulfilled by the graphs corresponding to triangulations. Hence it would be desirable to generate triangulations with weights such that other topological properties of graphs can be tuned.

In this paper we describe a mechanism that can be used to restrict the class of graphs occurring in the perturbation expansion according to topological properties.

This is achieved in two steps. First, we shall introduce topological actions for random graphs. This idea is closely related to a recent work by Dijkgraaf and Witten [13] and uses central extensions of groups. Next we show how these topological actions can be generated from twisted multimatrix models (a very brief and preliminary presentation without any proofs of the results has been given in [14]). All the matrix models under consideration will involve arbitrary $N \times N$ complex matrices with certain Hermiticity conditions. Although the described mechanism leads to topological actions for two-dimensional triangulations, the general idea might also be applicable for other cases.

Section 2 will review the notion of group extensions. Then we describe (section 3) the construction of topological actions on graphs using the cocycles of central extensions. The twisting of multimatrix models and hence the generating functional for statistical systems with topological actions on random graphs will be discussed in chapter 4. One main example will be the topological action induced by the Heisenberg group, i.e. central extensions of the group of two-dimensional translations. This example will be described in section 5. One of the possible applications of topological actions on graphs will be the cancellation of graphs with special properties. An example is discussed in section 5.

2. Group extensions

The construction of topological actions and twisted multimatrix models is based on central group extensions. We first review the basic structure and fix the notation: A group E is called an extension of a group G by a group H if there exists an exact sequence

$$1 \longrightarrow \mathrm{H} \longrightarrow \mathrm{E} \longrightarrow \mathrm{G} \longrightarrow 1. \tag{1}$$

This implies that H is a normal subgroup of E, i.e. the quotient E/H is isomorphic to a group G. We denote by $\alpha : \mathrm{G} \rightarrow \mathrm{E}$ a set of representatives of the equivalence classes of E. Any element $\gamma \in \mathrm{E}$ can be written as the representative $\alpha(g)$ of the equivalence class of γ multiplied by an element in H: $\gamma = h\alpha(g)$. Choosing such a decomposition for the elements in E one obtains for the product

$$\begin{aligned} h_1\alpha(g_1)h_2\alpha(g_2) &= h_1(\alpha(g_1)h_2\alpha(g_1)^{-1})\alpha(g_1)\alpha(g_2) \\ &= h_1 s_{g_1}(h_2)c(g_1,g_2)\alpha(g_1g_2). \end{aligned} \tag{2}$$

As H is a normal subgroup one has $s_g(h) := \alpha(g)h\alpha(g)^{-1} \in \mathrm{H}$. $c(g_1,g_2)$ defines a mapping from $\mathrm{G} \times \mathrm{G}$ into H. In the following we shall assume that the extension is central, i.e. H is Abelian and in the centre of E. This implies especially $s_g(h) = h$.

The associativity of the product in E leads to a (multiplicative) cocycle condition on $c(g_1,g_2)$:

$$c(g_1,g_2)c(g_1g_2,g_3) = c(g_1,g_2g_3)c(g_2,g_3) \tag{3}$$

and we shall refer to $c(g_1,g_2)$ as cocycles. The existence of non-trivial cocycles is a condition on the second cohomology group on G. It will turn out to be convenient to represent the elements in E by pairs of elements in H and G. The multiplication will be written as

$$(h_1,g_1)(h_2,g_2) = (h_1h_2c(g_1,g_2), g_1g_2) h_i, c(g_1,g_2) \in \mathrm{H} \qquad g_1,g_2 \in \mathrm{G}. \tag{4}$$

For most applications in the following we shall take G and H to be finite.

Choosing the identity element in E as the representative of the equivalence class of H implies

$$c(g, e_G) = c(e_G, g) = e_H \tag{5}$$

(e_H, e_G being the units in H and G respectively). If g and g^{-1} belong to different equivalence classes one can also require that $\alpha(g^{-1}) = \alpha(g)^{-1}$, i.e.

$$c(g, g^{-1}) = e_H. \tag{6}$$

This is in general not possible for involutions in G, i.e. elements for which $g^2 = e_G$. In the following we shall consider only central extensions for which condition (6) can be imposed.

Finally, we define a 'trace' mapping from E into the complex numbers by

$$\operatorname{tr}(h, g) = \hat{h}\delta(g) \tag{7}$$

where

$$\delta(g) = \begin{cases} 1 & \text{if } g = e_G \\ 0 & \text{else.} \end{cases} \tag{8}$$

The 'hat' denotes a one-dimensional complex representation of H.

3. Topological actions for statistical models on graphs

The cocycle condition (3) has a simple graphical representation (figure 1) which resembles the flip moves used in the generation of triangulations of two-dimensional surfaces [1, 2, 4] (figure 2). It is known that these moves are ergodic (at least for the planar case) [4], i.e. any two triangulations with the same number of triangles can be converted into each other by these flip moves. On the other hand, if the action of a statistical model on a graph—corresponding to a triangulation of a two-dimensional surface—happens to be invariant under these flip operations the corresponding partition function only depends on the topology of the surface, i.e. the Euler characteristic. Such an action is called a (two-dimensional) topological action.

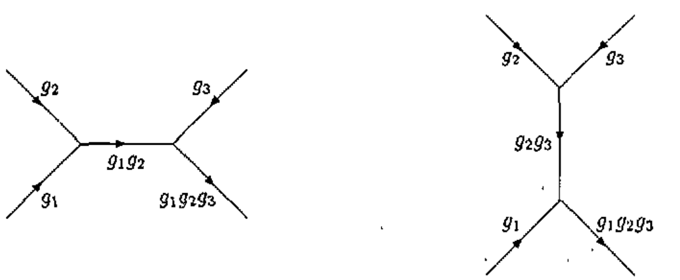

Figure 1. Graphical representation of the associativity condition for the cocycles.

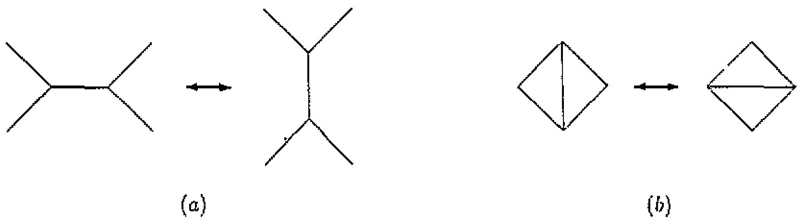

Figure 2. The flip to generate triangulations: (*a*) for the graphs of degree 3, (*b*) for the triangulation (dual of a graph of degree 3).

In this section we shall describe how the cocycles of central group extensions can be used to construct topological actions. First, we shall consider the case of (two-dimensional) triangulations, i.e. the dual graph is regular of degree three (each vertex has three hooks). Later we generalize these concepts to arbitrary graphs. It should be noted that the graphs under consideration have a natural cyclic ordering of the lines around each vertex. In matrix models—which will be the examples studied later—this ordering is inherent in the Feynman graphs generated (see e.g. [12]). For a given embedding of a graph into a two-dimensional surface this cyclic ordering is also fixed.

The statistical models with topological actions are defined as follows: degrees of freedom are the elements of a group G which are attached to the lines of a directed graph. This defines also group variables for the hooks at each vertex where we take the convention that for an outgoing line the inverse group element is to be taken. An allowed configuration is subject to the condition that at each vertex the product of the group variables should be the identity element. This definition resembles vertex models in the sense that degrees of freedom are attached to the lines of a graph with constraints at the vertices. On the dual graph (triangulation) the constraint becomes a zero curvature condition around each triangle [13].

Given a configuration of group elements on the lines of a graph satisfying the constraint one can attach the cocycle $c(g_1, g_2)$ of a central extension of G as a weight to each vertex (figure 3(*a*)), where g_1 and g_2 are two of the three group elements in cyclic order around this vertex. The cocycle condition (3) and the normalization (6) ensure that this assignment is cyclic:

$$c(g_1, g_2) = c(g_2, (g_1 g_2)^{-1}) = c((g_1 g_2)^{-1}, g_1). \tag{9}$$

The total weight for a configuration on a graph is the product of the cocycles (in a one-dimensional representation of H) at each vertex:

$$e^{i\pi S_{\mathrm{top}}(\{g\})} = \prod_{\text{vertices}\, v} \hat{c}(g_1^v, g_2^v). \tag{10}$$

This action is invariant under the flip moves because of the cocycle condition (3), so it depends only on the genus of the surface in which the graph is embedded. In principle it could also depend on the number of vertices as this is kept fixed under the flip moves. We shall show in a more general context later (see equation (14)) that this is actually not the case.

It is often convenient not to restrict oneself to regular graphs of degree 3. We now describe how one can construct a topological action for arbitrary graphs (embedded

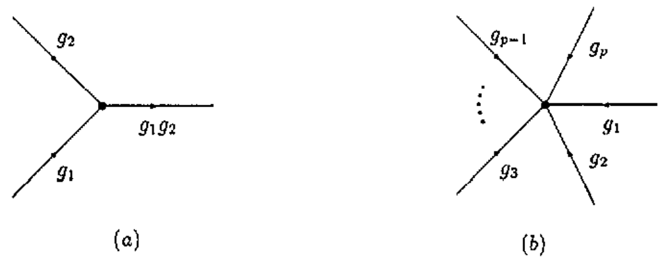

Figure 3. The assignment of group variables to the links around a vertex: (*a*) of degree 3, (*b*) of arbitrary degree.

without intersections of lines on a two-dimensional surface of sufficiently high genus). Again the statistical degrees of freedom are the group elements attached to the lines of a directed graph with the (zero curvature) constraint that the product of group elements (taking into account the direction of the lines) around each vertex is the identity element (figure 3(*b*)). If $g_1, g_2, \cdots, g_p$ are the group elements in cyclic order around a vertex, the weight attached to this vertex is

$$\begin{aligned}\omega(\{g^v\}) &= \mathrm{tr}[(e_{\mathrm{H}}, g_1)(e_{\mathrm{H}}, g_2)\cdots(e_{\mathrm{H}}, g_p)] \\ &= \hat{c}(g_1, g_2)\hat{c}(g_1g_2, g_3)\hat{c}(g_1g_2g_3, g_4)\cdots\hat{c}(\Pi_{i=1}^{p-1} g_i, g_p)\delta(\Pi_{i=1}^{p} g_i).\end{aligned} \tag{11}$$

Again, as the trace is cyclic, this expression does not depend on the choice of which line is taken to be the first. The topological action for a configuration is given by the product of the local weights:

$$\mathrm{e}^{\mathrm{i}\pi S_{\mathrm{top}}(\{g\})} = \prod_{\text{vertices}\, v} \omega(\{g^v\}). \tag{12}$$

The partition function is obtained by taking the sum over all possible configurations on the graph:

$$Z = \sum_{\{g\}} \mathrm{e}^{\mathrm{i}\pi S_{\mathrm{top}}(\{g\})}. \tag{13}$$

For finite central extensions the one-dimensional complex representations of the cocycles are phases, hence the action $S_{\mathrm{top}}(\{g\})$ is real.

The definition of topological weights for graphs with vertices of arbitrary degree together with the normalization of the cocycles (5), (6) and the trace (7), make the topological action invariant under a broader class of changes of the graph:

(i) Insertion or deletion of a line connecting two different vertices (figure 4). This property is the graphical equivalent of

$$\begin{aligned}&\mathrm{tr}[(e_{\mathrm{H}}, g_1)\cdots(e_{\mathrm{H}}, g_{n-1})(e_{\mathrm{H}}, g_n)]\,\mathrm{tr}[(e_{\mathrm{H}}, g_n^{-1})(e_{\mathrm{H}}, g_{n+1})\cdots(e_{\mathrm{H}}, g_{n+m})] \\ &\qquad = \delta(\Pi_{i=1}^{n} g_i)\,\mathrm{tr}[(e_{\mathrm{H}}, g_1)\cdots(e_{\mathrm{H}}, g_{n-1})(e_{\mathrm{H}}, g_{n+1})\cdots(e_{\mathrm{H}}, g_{n+m})]\end{aligned} \tag{14}$$

which follows from

$$\begin{aligned}&\mathrm{tr}[(h_1, g_1)(e_{\mathrm{H}}, g)]\,\mathrm{tr}[(e_{\mathrm{H}}, g^{-1})(h_2, g_2)] \\ &\qquad = \delta(g_1g)\delta(g^{-1}g_2)\hat{h}_1\hat{h}_2 = \delta(g_1g)\,\mathrm{tr}[(h_1, g_1)(h_2, g_2)].\end{aligned} \tag{15}$$

No cocycle factors appear, because of (6). This invariance of the topological action holds only for allowed configurations which satisfy the constraints.

(ii) Insertion or deletion of a self-loop connecting adjacent hooks at a vertex (figure 5). This is also a consequence of (6) as

$$(e_{\mathrm{H}}, g)(e_{\mathrm{H}}, g^{-1}) = (e_{\mathrm{H}}, e_{\mathrm{G}}). \tag{16}$$

Furthermore, for a representation of a graph in a plane (which might have intersections of lines) we have the invariance:

(iii) Moving a line through a vertex in a planar representation of the graph (figure 6). This is a trivial consequence of the fact that the product of the cocycles does not depend on this representation.

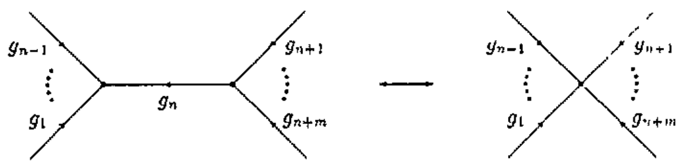

Figure 4. Deletion and insertion of lines connecting two different vertices.

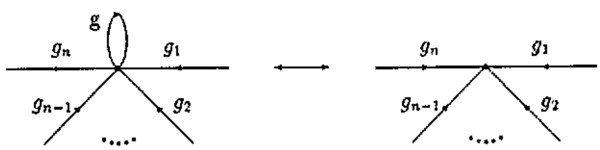

Figure 5. Deletion and insertion of a self-loop without intersections.

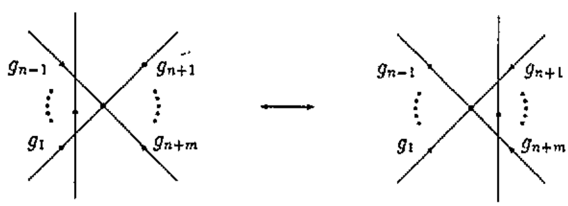

Figure 6. Moving a line through a vertex in a planar representation of the graph.

The first of these moves allows reduction of the graph successively to a graph consisting of one vertex only. The second move can be used to eliminate all self-loops of this one-point graph which connect neighbouring hooks. The product of cocycles essentially depends only on the intersections of the remaining lines. As a trivial consequence one obtains that the weight for planar diagrams without external lines is always one.

In later applications (see the discussion of the dimer and vertex model) we will consider statistical models defined on random graphs where not every vertex type is allowed but only certain classes. In some cases the constraint conditions for these models are less restrictive and admit groups with central extensions which in the general case might not exist. For such models the insertion of a line is not, in general, possible as it might lead to vertex types which are not allowed. If, for instance, the general vertex of degree 3 does not exist the invariance under flips is no longer guaranteed. In these cases topological properties of the embedding of a graph into a surface can be important.

The notion of 'topological action' will also be used for actions which are only invariant under the restricted class of moves. However, even in cases where the insertion of lines will, in general, not be possible, the deletion of lines is always possible, so that the method of reducing the graph to a one-point graph helps to calculate the product of cocycles.

4. Multimatrix models leading to topological actions

Having fixed the notation for central extensions and elaborated the properties of cocycles leading to topological actions on graphs, we now describe the general construction of the twisted matrix models. These are multimatrix models for which the perturbation expansion generates statistical systems on random graphs with topological weights corresponding to the products of cocycles.

Given a multimatrix model, one labels the matrices by elements of a group G such that

$$M^{+}(g) = M(g^{-1}) \tag{17}$$

and the action becomes

$$S[\{M(g)\}] = \sum_{g} \lambda(g) \operatorname{tr} M(g^{-1})M(g) + \sum_{\{g_i\}} f(g_1, g_2, \ldots)\delta(\Pi_i g_i) \operatorname{tr}[\Pi_i M(g_i)] \tag{18}$$

where $\lambda(g)$ and $f(g_1, \ldots)$ are arbitrary couplings. The non-trivial part is to find a group G such that the 'momentum' constraint $\prod_i g_i = e_{\mathrm{G}}$ holds for every non-vanishing term in the interaction. This might be possible only after a linear transformation on the set of the matrices (see the discussion of the vertex model). Suppose one can find such a labelling by a group G which admits a central extension, then one obtains the twisted matrix model by the following replacement in (18):

$$\delta(\Pi_i g_i) \longrightarrow \operatorname{tr}[(e_{\mathrm{H}}, g_1)(e_{\mathrm{H}}, g_2)\cdots] = \delta(\Pi_i g_i)\hat{c}(g_1, g_2)\hat{c}(g_1 g_2, g_3)\cdots. \tag{19}$$

Essentially, the coupling constants get multiplied by the product of cocycle factors for each vertex type.

The perturbation expansion for the free energy F per degree of freedom of the matrix integral

$$Z = \mathrm{e}^{N^2 F} = \int \prod_{g} \mathrm{d}M(g) \exp(-S[\{M(g)\}]) \tag{20}$$

leads to a summation over connected graphs, where the contribution for each graph can be interpreted as a generalized vertex model (degrees of freedom $\{g\}$ attached to the lines with a 'momentum' constraint at each vertex). The statistical weight for this model (considered as a function of the configuration $\{g\}$) depends on the coupling constants $f(\{g_i\}), \lambda(g_i)$, as in the untwisted model, and on a term containing products of the cocycles.

The product of the cocycles adds a topological action to the statistical model on the random graphs exactly of the type of topological actions discussed in section 4.

5. Examples and applications

One case can be treated in general and is relatively trivial: if the central extension is an Abelian group, the topological action is always 0 for graphs without external lines. This follows as the commutativity allows the change of the order of outgoing hooks at vertices. Together with the moves described in section 3 which leave the topological action invariant this allows the reduction of any graph without external lines to a planar graph and finally to one point.

One of the most well-known non-trivial examples for a central extension is the Heisenberg group. Taking for G the group of two-dimensional translations $\mathbb{R}^2$ and for H the multiplicative group $\mathbb{C}-\{0\}$, one obtains a central extension from the composition rule

$$(z_1, a)(z_2, b) = (z_1 z_2 q^{a_1 b_2 - a_2 b_1}, a + b) \tag{21}$$

where, in general, q is an arbitrary, fixed, non-vanishing complex number. It is easy to verify that this q-dependent factor satisfies the cocycle condition (3). Finite dimensional representations exist for the induced extensions of translations on a periodic $\mathbb{Z}_N \times \mathbb{Z}_N$ lattice. In these cases q is a root of unity. Explicit representations can be obtained from the following $N \times N$ matrices:

$$Q = \begin{pmatrix} \omega & 0 & 0 & \cdots & 0 \\ 0 & \omega^2 & 0 & \cdots & 0 \\ 0 & 0 & \omega^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^N = 1 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{22}$$

where

$$\omega = \mathrm{e}^{2\pi \mathrm{i}/N}. \tag{23}$$

They satisfy the relations

$$P^N = Q^N = \mathbf{1} \qquad PQ = \omega QP. \tag{24}$$

For the matrices

$$T(m, n) = \mathrm{e}^{\mathrm{i}\pi mn/N} Q^m P^n \tag{25}$$

this implies the product rule

$$T(m_1, n_1)T(m_2, n_2) = \mathrm{e}^{-(\mathrm{i}\pi/N)(m_1 n_2 - m_2 n_1)} T(m_1 + m_2, n_1 + n_2) \tag{26}$$

i.e. they form a representation of the above-mentioned extensions of $\mathbb{Z}_N \times \mathbb{Z}_N$. Further relations are

$$T^{+}(m,n) = T(-n,-m) \tag{27}$$

$$\operatorname{tr} T(m,n) = N\delta(m,n)(\delta(m,n) := \delta_{m0}\delta_{n0}). \tag{28}$$

The normalized trace is

$$\operatorname{tr} T(m,n) := \frac{1}{N} \operatorname{tr} T(m,n) = \delta(m,n). \tag{29}$$

For these central extensions one can calculate the topological action, i.e. the product of cocycle factors, in a closed form for an arbitrary graph [14]. Let $\{(m_i, n_i)\}$ be the set of 'momenta' attached to the lines of a (directed) graph satisfying momentum conservation at each vertex and let

$$I_{ij} = \begin{cases} 1 & \text{line } j \text{ crosses line } i \text{ from the right} \\ -1 & \text{line } j \text{ crosses line } i \text{ from the left} \\ 0 & \text{otherwise} \end{cases}$$

be the intersection matrix for a planar representation of this graph, then one obtains

$$\mathrm{e}^{\mathrm{i}\pi S_{\mathrm{top}}[\{(m_i,n_i)\}]} = \exp\left(-\frac{2\pi\mathrm{i}}{N}\sum_{i,j} m_i I_{ij} n_j\right). \tag{30}$$

The proof follows immediately from the explicit form of the cocycles of the Heisenberg group after the graph has been reduced to a one-point graph. The topological properties of the cocycles of the Heisenberg group have been used implicitly in the theory of twisted reduced large N models [16, 17]. This result will now be used to discuss some examples.

5.1. A dimer model

We consider a Hermitian two-matrix model with the following action

$$S_D = \operatorname{tr} M_1^2 + \operatorname{tr} M_2^2 + \mu_1 \operatorname{tr} M_1^4 + \mu_2 \operatorname{tr} M_2^4 + \mu_3 \operatorname{tr} M_1^2 M_2^2 + \mu_4 \operatorname{tr}(M_1 M_2)^2 \tag{31}$$

corresponding to the four types of vertices:

This model describes polygons on random graphs (regular of degree 4). The different couplings weight the length of the polygon, bendings and the touching of two polygons. For the special case $\mu_1{:}\mu_2{:}\mu_3{:}\mu_4 = \mu^4{:}1{:}4\mu^2{:}2\mu^2$ one recovers the Ising model represented as closed polygons (see e.g. [18]).

To obtain a twisted version of this model one has to find a group generated by two elements g_1 and g_2, satisfying the following relations:

$$g_1^2 = g_2^2 = (g_1 g_2)^2 = e_G. \tag{32}$$

The first two of these relations are required by the quadratic part of the action. They also imply the constraints for the first three types of vertices. The last relation is required by the constraint for vertex type 4.

One solution is $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with (multiplicative notation) $g_1 = (-1, 1)$ and $g_2 = (1, -1)$. This group has a central extension by $H = \{+1, -1\}$ generated by the Pauli matrices σ_1 and σ_3. Following the procedure described in section 4, equation (19), one obtains a sign flip for μ_4, i.e. one can introduce an effective topological action which counts the number of configurations of type 4. The theorem that the topological weight is 1 for planar graphs states that this number is always even on planar graphs.

As an example of how statistical models with topological actions can lead to the cancellation of graphs with certain topological properties, we consider the special case of couplings $\mu_1 : \mu_2 : \mu_3 : \mu_4 = 1:1:4:2$. (This case corresponds to the twisted Ising model at infinite temperature which is equal to the complex matrix model.) One can calculate the partition function for each graph exactly [14]: it is non-zero only for those graphs, for which the one-point reduced graph in a planar representation has the property that each line has an even number of intersections. (This characteristic is independent of the representation chosen.)

This result is an example for the general remark in section 3: topological actions can depend on the kind of embedding of a graph into a two-dimensional surface, in this case the property that for the one-point reduced graph in a planar representation each line has an even number of intersections. The dimer model (31) is formulated for vertices of degree 4 and the contraction of lines always leads to vertices of even degree. On the other hand, the invariance of the action under the splitting of a vertex and insertion of a line holds only, if each of the two resulting vertices have even degree. The constraint condition for vertices of odd degree cannot be solved with g_1 and g_2 only.

As a corollary one obtains that the moves of inserting or deleting lines and self-loops are not ergodic on non-planar graphs if restricted to vertices of even degree. To change the property 'each line in the planar representation of the one-point reduced graph has an even number of intersections' one has to split vertices of even degree into vertices of odd degree.

5.2. *A 'four-vertex' model*

The statistical model corresponding to the matrix action (31) looks almost like an 'eight'-vertex model on random graphs, where it has to be noted that among the eight possible spin configurations of the eight-vertex model [19] the distinction of only the four following configurations makes sense on random (four valence) graphs:

As incoming spins have to combine with outgoing spins and *vice versa* the action leading to the 'four' vertex model is

$$S_4 = \operatorname{tr} M_1 M_2 + \lambda_1 (\operatorname{tr} M_1^4 + \operatorname{tr} M_2^4) + \lambda_2 \operatorname{tr} M_1^2 M_2^2 + \lambda_3 \operatorname{tr} (M_1 M_2)^2. \tag{33}$$

Vertex types 1 and 2 get the same weight, as this can always be achieved by a rescaling $M_1 \to \lambda M_1, M_2 \to 1/\lambda M_2$. (For closed regular graphs of degree 4 the number of vertex types 1 and 2 always has to be equal, hence only the product of their weights enters into the partition sum.) At first sight it looks as if this model does not have a non-trivial twisted version: the constraint equations for the group generators are

$$g_1 g_2 = 1 \qquad \text{and} \qquad g_1^4 = 1 \tag{34}$$

with solutions $\mathbb{Z}_4$ or $\mathbb{Z}_2$. The central extensions of these groups are always Abelian. However, the linear transformation

$$M_1 = (\tilde{M}_1 + \tilde{M}_2) \qquad M_2 = (\tilde{M}_1 - \tilde{M}_2) \tag{35}$$

brings (33) into the form

$$S_4 = \operatorname{tr} \tilde{M}_1^2 - \operatorname{tr} \tilde{M}_2^2 + \tilde{\mu}_1(\operatorname{tr} \tilde{M}_1^4 + \operatorname{tr} \tilde{M}_2^4) + \tilde{\mu}_2 \operatorname{tr} \tilde{M}_1^2 \tilde{M}_2^2 + \tilde{\mu}_3 \operatorname{tr}(\tilde{M}_1 \tilde{M}_2)^2 \tag{36}$$

where the couplings are related as

$$\begin{aligned} \lambda_1 &= \tfrac{1}{8}\tilde{\mu}_1 + \tfrac{1}{16}\tilde{\mu}_2 + \tfrac{1}{16}\tilde{\mu}_3 \\ \lambda_2 &= \tfrac{1}{2}\tilde{\mu}_1 - \tfrac{1}{4}\tilde{\mu}_3 \\ \lambda_3 &= \tfrac{1}{4}\tilde{\mu}_1 - \tfrac{1}{8}\tilde{\mu}_2 + \tfrac{1}{8}\tilde{\mu}_3. \end{aligned} \tag{37}$$

This model admits a twisted version of the same type as the dimer model (31). The fact that one obtains a twisted model only after a linear transformation of the matrices reveals that one is actually using central extensions of the group algebra.

5.3. The Hermitian matrix model

It might be interesting to note that the Hermitian matrix model also follows from a twisting procedure as has been described in general in section 4. One starts with a model of N^2 copies of a simple integral in one real variable:

$$Z = \left(\int \mathrm{d}\varphi \, \mathrm{e}^{(-\varphi^2 - g\varphi^p)} \right)^{N^2}. \tag{38}$$

This can be rewritten as an $N \times N$ 'lattice' model without any interactions between fields at different sites:

$$Z = \int \prod_{x,y=1}^{N} \mathrm{d}\varphi_{x,y} \exp\left(-\sum_{x,y} \varphi_{x,y}^2 - g \sum_{x,y} \varphi_{x,y}^p \right). \tag{39}$$

A Fourier transformation for these fields,

$$\varphi_{x,y} = \frac{1}{N} \sum_{(m,n)} \tilde{\varphi}(m,n) \mathrm{e}^{(2\pi \mathrm{i}/N)(mx+ny)} \tag{40}$$

leads to an action

$$S = \sum_{(m,n)} \tilde{\varphi}(-m,-n)\tilde{\varphi}(m,n) + \frac{g}{N^{p-2}} \sum_{\{(m_i,n_i)\}} \delta(\Sigma_i (m_i, n_i)) \prod_i \tilde{\varphi}(m_i, n_i). \quad (41)$$

This is of the required type: the index set has the structure of a group ($\mathbb{Z}_N \times \mathbb{Z}_N$), and the momentum constraint holds at each vertex. The N-dimensional representations (25) are a central extension of this group. Replacing the δ-function in (41) by the normalized trace of products of T-matrices (25), one obtains

$$S = -\operatorname{tr} M^2 - \frac{g}{N^{p/2-1}} \operatorname{tr} M^p \quad (42)$$

with

$$M = \frac{1}{N^{1/2}} \sum_{(m,n)} \tilde{\varphi}(m,n) T(m,n). \quad (43)$$

As any Hermitian matrix can be decomposed in this way (the Hermiticity condition for M is the same as the reality condition for the Fourier modes $\tilde{\varphi}$), the twisting leads to the standard Hermitian matrix model. Even the correct powers of $N^{-(1/2)}$ occur in the coupling to ensure a non-trivial $N \to \infty$ limit.

While the untwisted model (38) can only distinguish the number of vertices of graphs by power counting of g, the matrix model is also known to distinguish the genus of a graph by power counting of N. This result is recovered if one calculates the (normalized) partition function for the topological action (30) on an arbitrary graph [15]:

$$\frac{1}{\sum_{\{m_i,n_i\}=1}^{N} 1} \sum_{\{m_i,n_i\}=1}^{N} \exp -\frac{2\pi i}{N} \Sigma_{i,j}^{L} m_i I_{ij} n_j = \frac{1}{N^{\operatorname{rank}(I)}}. \quad (44)$$

As the rank of the intersection matrix I is twice the genus of the Feynman graph one obtains the standard result.

6. Conclusions

It is shown how matrix models can be used to generate topological actions for graphs. These actions are related to central extensions of groups which serve as index sets for the matrices. Restricting the possible structure of vertices allows the construction of more general topological actions which also distinguish classes of embeddings of the graphs in two-dimensional surfaces.

A natural question which arises is the matter content of the twisted matrix models. This is currently under investigation. The given examples, especially the representation of the standard matrix model as a twisted model and the twisted version of the vertex models, might be helpful in clarifying this problem.

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